International Journal of Theoretical Physics, Vol. 14, No. 4 (1975), pp. 277–288

Wave Functions Relative to a Real Polarization*

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Received: 14 *March* 1975

Abstract

The structure of the space of wave functions in the representation given by a complete everywhere independent set of commuting observables is analyzed in the framework of geometric quantization. Under the assumptions that the chosen real polarization of the classical phase space is locally trivial and complete, it is shown that the wave functions are generalized sections of an appropriate line bundle with supports determined by generalized Bohr–Sommerfeld conditions. There is a canonical Hilbert subspace of the space of the wave functions with the scalar product defined in terms of the same expressions which appear in the generalized Bohr–Sommerfeld conditions.

1. *Introduction*

In the abstract formulation of quantum mechanics, states are elements of an abstract Hilbert space. A choice of a complete system of commuting observables yields a representation of states by wave functions. Knowledge of the classical counterparts of the observables forming the complete commuting system enables one to interpret the wave functions from the point of view of the classical phase space. However, in the process of quantization of a classical system the situation is reversed. One has only the classical phase space to begin with, and one has to choose a maximal set of commuting functions on the phase space to define wave functions and their scalar products. The standard quantization procedure is possible if the classical system has a distinguished configuration space with a Euclidean structure. Then, one uses Cartesian coordinates of the configuration space as the complete set of commuting observables, and the wave functions form the space of square integrable complex functions on the configuration space. In the case where there is no Euclidean structure on the configuration space the quantization procedure is more difficult and the geometric nature of wave functions is more complicated. If some of the observables in the complete commuting set have discrete spectra, the wave functions are in fact generalized functions (distributions) on the phase space and one has to be very careful in

* Partially supported by National Research Council of Canada Grant No. A8091.

defining their scalar product, since multiplication of generalized functions is usually not possible.

The aim of this paper is to analyze the space of wave functions in the representation given by a complete everywhere independent set of commuting observables without imposing any additional structure on the classical phase space. It is done here in the framework of geometric quantization theory introduced by Kostant (1970), and developed and applied to problems in group representation and quantum mechanics by several authors (cf. Auslander & Kostant, 1971; Blattner, 1973, 1974; Kostant, 1973, 1974; Gawedzki, 1974; Rawnsley, 1974; Simms, 1973, 1974a, b; Śniatycki, 1974, 1975). Some of the essential ideas in geometric quantization were developed independently by Souriau (1970).

The phase space of a dynamical system with n degrees of freedom can be represented by a $2n$-dimensional manifold X. The Lagrange bracket divided by the Planck's constant defines a symplectic form ω on X. Dynamical variables are represented by functions on X.

A symplectic manifold (X, ω) is quantizable if ω defines an integral de Rham cohomology class. In this case there exists a complex line bundle L with connection ∇ such that ω is the curvature form of ∇, and an invariant Hermitian form. Given such a line bundle, one can associate to each function on X a linear operator on the space of sections of L in such a way that the Poisson bracket of two functions is associated to the commutator of the corresponding operators, divided by $i\hbar$, where $\hbar$ is the Planck's constant divided by 2π, see Kostant (1970). In order to obtain physically meaningful quantization one has to choose a complete system of commuting observables. Since one has no Hilbert space of states, one does not know a priori which functions on X will qualify as observables. A globalized classical counterpart of the notion of a complete set of everywhere independent observables is that of a real polarization of a symplectic manifold. A real polarization of (X, ω) is a foliation F of X by Lagrangian submanifolds, that is by n-dimensional submanifolds Q of X, called leaves of the foliation, such that ω restricted to Q vanishes identically. The sections of L covariant constant along F are possible candidates for wave functions. However, there is no natural way to define a scalar product for such sections, since there is no canonical density in the space of all leaves of the foliation F. This is one of the reasons for the necessity for introducing a bundle $N^\dagger$ of half-forms relative to F and defining the wave functions to be the sections of $L \otimes N^\dagger$ covariant constant along F. Such smooth sections exist only if the leaves of F are simply connected. A complete study of this case is given in Blattner (1973). Sections of $L \otimes N^\dagger$ can be treated also as generalized sections of a bundle $L \otimes N$, where N is the Hermitian dual of $N^\dagger$. Therefore, wave functions can be represented by generalized sections of $L \otimes N$ covariant constant along F. This interpretation of the wave functions can be used also in the case when the leaves of F are not simply connected. Throughout this paper we assume some topological conditions on the allowable polarizations, namely local triviality and completeness, which imply that leaves of F are diffeomorphic to $T^k \times R^{n-k}$, where T^k denotes a k torus. It is shown that the space of generalized sections of $L \otimes N^{\dagger}$ covariant constant along F has a canonically defined pre-Hilbert space H_0. The completion H of H_0 is defined to be the Hilbert space of wave functions in the representation given by polarization F. Quantization of functions constant along F gives operators of multiplication by these functions, and the spectra of these operators are completely determined by the generalized Bohr–Sommerfeld conditions, cf. Śniatycki (1975). In order to quantize arbitrary functions one would have to generalize Blattner–Kostant–Sternberg kernels (Blattner, 1973, 1974) to the case of polarizations with not simply connected leaves. This problem will be studied separately.

In the following, all manifolds are real, finite-dimensional, C^∞, Hausdorff, and paracompact, and all functions, maps, etc. are C^∞ (smooth) unless otherwise specified. Most of the results in differential geometry used in this paper can be found in Kobayashi & Nomizu (1963).

2. *Complete Locally Trivial Real Polarizations*

Let (X, ω) be a symplectic manifold and $\dim X = 2n$. A real polarization of (X, ω) is an integrable n-dimensional distribution F on X such that ω restricted to vectors in F vanishes identically. For each point $x \in X$ there exists a unique integral manifold Q_x of F passing through x such that $\omega \mid Q_x = 0$. Therefore F defines a foliation of X by n-dimensional manifolds, called the leaves of the foliation. We shall use the same symbol F to denote the foliation defined by a real polarization F. Locally leaves of F can be characterized by n equations $f_1 = \text{const}, \ldots, f_n = \text{const}$, where $f_1, \ldots, f_n$ are independent functions defined on an open subset V of X. For each function f on X the Hamiltonian vector field ξ_f of f is defined by $\xi_f \lrcorner \omega = df$, where $\lrcorner$ denotes the left-invariant product of a form by a vector field, satisfying $(\xi_1 \lrcorner \omega)(\xi_2) = \omega(\xi_1, \xi_2)$ for any vector fields ξ_1, ξ_2. The Hamiltonian vector fields $\xi_{f_1}, \ldots, \xi_{f_n}$ of $f_1, \ldots, f_n$ have values in F and therefore $\omega(\xi_{f_i}, \xi_{f_j}) = 0$, for all $i, j = 1, \ldots, n$. This implies that the Poisson brackets of the functions $f_1, \ldots, f_n$ are identically zero, since the Poisson bracket of f_i and f_j is proportional to $\omega(\xi_{f_i}, \xi_{f_j})$. Hence, the notion of a real polarization is a globalization of the notion of a complete set of everywhere commuting functions.

Definition 2.1. A real polarization F of a symplectic manifold (X, ω) is locally trivial if the set Y of all leaves of F has a manifold structure such that the canonical projection $\eta: X \to Y$, assigning to each $x \in X$ the leaf of F containing x, is a locally trivial fibration.

Let F be a locally trivial real polarization of (X, ω) and U a coordinate neighborhood in Y with coordinate functions $g_1, \ldots, g_n$. For each $i = 1, \ldots, n$, we denote by ξ_i the Hamiltonian vector field of $f_i = g_i \circ \eta$ on $\eta^{-1}(U) \subset X$. The Hamiltonian vector fields $\xi_1, \ldots, \xi_n$ commute and span $F \mid \eta^{-1}(U)$. Therefore, for each leaf Q of F contained in $\eta^{-1}(U)$, the restrictions of $\xi_1, \ldots, \xi_n$ to Q, denoted by $\xi_1 \mid Q, \ldots, \xi_n \mid Q$, commute and span TQ. This defines an absolute parallelism in Q. A vector field on Q is parallel if its components with respect to the basis $(\xi_1 \mid Q, \ldots, \xi_n \mid Q)$ are constant. The absolute parallelism defined in this way in Q is independent of the choice of Hamiltonian vector fields spanning F. This is a special case of the results of Weinstein (1971).

Definition 2.2. A locally trivial polarization F is complete if, for each leaf Q of F, every parallel vector field on Q is complete.

Examples of locally trivial real polarizations are furnished by the Schrödinger representation, in which X is identified with the cotangent bundle space T^*Y of the configuration manifold Y, and by quantization of a one-dimensional harmonic oscillator in the representation in which the Hamiltonian is diagonal (Simms, 1973). Moreover, it has been shown by Rawnsley (unpublished) that quantization of a one-dimensional harmonic oscillator in the polarization given by rays starting at the origin gives the wrong energy spectrum; this polarization is not complete. The condition that a locally trivial real polarization should be complete is related to the Pukansky condition in Auslander and Kostant (1971).

A Euclidean cylinder is the quotient space of R^n by a properly discontinuous subgroup G of R^n generated by a set $(u_1, \ldots, u_k)$ of linearly independent vectors in R^n. The canonical metric in R^n induces in R^n/G a flat Riemannian metric.

Lemma 2.1. Each leaf of a complete locally trivial real polarization is diffeomorphic to a Euclidean cylinder.

Proof. The proof is based on results in Auslander & Markus (1955) and Kobayashi & Nomizu (1963, Chap. V). Each leaf Q of a complete locally trivial polarization has a complete flat affine connection defined by absolute parallelism. Hence the universal covering space of Q is isomorphic to R^n. Introducing a Riemannian metric in Q such that the basic parallel vector fields $\xi_1|Q, \ldots, \xi_n|Q$ are orthonormal, and choosing a base point $x \in Q$, induces in Q a structure of an Abelian group with an invariant flat metric such that the covering map $R^n \to Q$ is a group homomorphism. Hence, $Q \simeq R^n/G$, where G is a subgroup of R^n generated by $k \leqslant n$ linearly independent vectors in R^n.

Lemma 2.2. Let F be a complete locally trivial real polarization of (X, ω) and R^n/G a Euclidean cylinder diffeomorphic to a typical leaf of F. For each $y \in Y$, there exist a neighborhood U of y and a trivialization φ: $\eta^{-1}(U) \to U \times R^n/G$ such that, for each leaf Q of F contained in $\eta^{-1}(U)$, the induced diffeomorphism φ_Q: $Q \to R^n/G$ is an isomorphism of manifolds with absolute parallelism.

Proof. Let U be a coordinate neighborhood of y in Y such that $\eta^{-1}(U)$ is trivial and $\xi_1, \ldots, \xi_n$ the Hamiltonian vector fields on $\eta^{-1}(U)$ defined by the coordinate functions on U. Since $\eta^{-1}(U)$ is trivial there exists a section s: $U \to X$ of η. For each leaf Q in $\eta^{-1}(U)$, the section s defines a base point in Q. Introducing in Q a flat Riemannian metric such that $\xi_1|Q, \ldots, \xi_n|Q$ are orthonormal vector fields defines in Q a structure of an Abelian group and an isomorphism of Lie groups φ_Q: $Q \to R^n/G$. The isomorphism φ_Q maps the vector fields $\xi_1|Q, \ldots, \xi_n|Q$ to the parallel vector fields on R^n/G defined by the coordinate directions in R^n. Hence, φ_Q is an isomorphism of manifolds with absolute parallelism. Moreover, the isomorphism φ_Q is uniquely determined by the base point $s(U) \cap Q$ and the restriction to Q of the vector fields $\xi_1, \ldots, \xi_n$. Therefore, the map φ: $\eta^{-1}(U) \to U \times R^n/G$ given by $\varphi(x) = (\eta(x), \varphi_Q(x))$, where Q is the integral manifold of F through x, is well defined, and it is a smooth trivialization satisfying the conditions of the Lemma.

The fundamental group of a Euclidean cylinder R^n/G is isomorphic to G, which is a free Abelian group of rank $k \leqslant n$. Therefore, for each leaf Q of F and each $x \in Q$, the fundamental group of Q with base point x is a free Abelian group of rank k. For any $x, x' \in Q$, there is a natural isomorphism of fundamental groups of Q with base points x and x', respectively. We shall use these isomorphisms to identify fundamental groups of Q with different base points and denote by $\pi_1(Q)$ the fundamental group of Q obtained by this identification. For any diffeomorphism φ_Q: $Q \to R^n/G$ the induced isomorphism of the fundamental groups will be denoted by $\varphi_{Q*}: \pi_1(Q) \to G$.

Theorem 2.1. Let φ: $\eta^{-1}(U) \to R^n/G$ be a trivialization of η satisfying the conditions of Lemma 2.2. For each $u \in G$, there exists a vector field $\tilde{\zeta}_u$ in $F/\eta^{-1}(U)$ such that, for each $x \in \eta^{-1}(U)$, the integral curve $c: [0, 1] \to X$ of ζ_u originating at x is a closed loop in the leaf Q of F passing through x representing the element $\varphi_{Q*}^{-1}(u)$ in $\pi_1(Q)$. The collection of local vector fields on X defined in this way spans an involutive distribution K contained in F of dimension k equal to the rank of the fundamental group of a typical integral manifold of F. There is a canonically defined density κ on K.

Proof. Let $\tilde{\zeta}_u$ be a vector field on $U \times R^n/G$ defined by $\tilde{\zeta}_u(y, p) = (0, u)$ for each $(y, p) \in U \times R^n/G$. For anyintegral curve c: $[0, 1] \to U \times R^n/G$ of $\tilde{\zeta}_u$, the projection of c to U is a constant curve in U while the projection of c to R^n/G is a closed curve representing the element u of the fundamental group of R^n/G. Since φ: $\eta^{-1}(U) \to U \times R^n/G$ is a diffeomorphism such that, for each leaf Q of F in $\eta^{-1}(U)$, it induces an isomorphism $\varphi_Q: Q \to R^n/G$ of manifolds with absolute parallelism, the vector field ζ_u on $\eta^{-1}(U)$, defined by $\zeta_u(x) = T\varphi^{-1}(\tilde{\zeta}_u(\varphi(x)))$ satisfies the conditions of the Theorem. Further, the vector fields $\zeta_{u_1}, \ldots, \zeta_{u_k}$ corresponding to the generators of G span a k-dimensional involutive distribution on $\eta^{-1}(U)$ contained in $F|\eta^{-1}(U)$. A different choice of trivializations of $\eta^{-1}(U)$ satisfying the conditions of Lemma 2.2 leads to another k-tuple $(\zeta'_{u_1}, \ldots, \zeta'_{u_k})$ of linearly independent vector fields in $F|\eta^{-1}(U)$ which differs from $(\zeta_{u_1}, \ldots, \zeta_{u_k})$ by multiplication by a matrix with determinant ± 1. Therefore, local vector fields corresponding in this way to elements of G define a k-dimensional involutive distribution K on X contained in F, and a density κ on K.

3. *Complex Line Bundle*

Let L denote a complex line bundle over X with a connection ∇ such that ω is the curvature form of ∇, and an invariant Hermitian form $\langle \mid \rangle$. Such a line bundle exists if and only if ω defines an integral de Rham cohomology class; for a detailed exposition of the theory of complex line bundles see Kostant (1970). We denote by L_* the bundle L minus the zero section. It is a principal fiber bundle over X with the multiplicative group C_* of nonzero complex numbers as the structure group. The line bundle L is a fiber bundle associated to L_* with a typical fiber C on which C_* acts by multiplication. The connection ∇ in L corresponds to the horizontal distribution in L_*. Any piecewise smooth curve in X can be lifted to a horizontal curve in L_*. Any loop c: $[0, 1] \to X$ defines a unique complex number $\alpha_L(c) \in C_*$ such that, for each horizontal lift $\tilde{c}$: $[0, 1] \to L_*$ of c, $\tilde{c}(1) = \alpha_L(c)\tilde{c}(0)$. The function α_L from the space of all loops in X to C_* is called the scalar parallel transport function of the connection in L_*. It satisfies the following conditions:

1. $|\alpha_L(c)| = 1$ for each loop c in X.
2. If c_1 is homotopic to c_2 and Σ is a surface of deformation of c_1 to c_2, then
$$\alpha_L(c_1) = \exp\left(-2\pi i \int_\Sigma \omega\right)\alpha_L(c_2)$$
3. If $c_1(1) = c_2(0)$, then $\alpha_L(c_1 \cdot c_2) = \alpha_L(c_1)\alpha_L(c_2)$, where $c_1 \cdot c_2(t) = c_1(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $c_1 \cdot c_2(t) = c_2(2t - 1)$ for $\frac{1}{2} \leqslant t \leqslant 1$.

Let Q be a leaf of F. Since $\omega | Q = 0$, $L_* | Q$ has a flat connection, and the restriction of α_L to loops in Q depends only on the homotopy classes of loops in Q. This defines a homomorphism $\alpha_{L|Q}$: $\pi_1(Q) \to C_*$ from the fundamental group of Q to the group C_*.

4. *Metalinear Structures*

Let BF denote the bundle of ordered bases of F. It is a $GL(n, R)$ principal fiber bundle over X. Let $ML(n, C)$ denote the double covering of $GL(n, C)$. Since $GL(n, R)$ is a subgroup of $GL(n, C)$ the inverse image of $GL(n, R)$ under the covering map $\tilde{\rho}$: $ML(n, C) \to GL(n, C)$ is a subgroup $ML(n, R)$ of $ML(n, C)$ called the real $n \times n$ metalinear group.

Let $\tilde{\chi}$: $ML(n, C) \to C$ be the unique holomorphic square root of the complex character Det o $\tilde{\rho}$ of $ML(n, C)$ such that $\tilde{\chi}(1) = 1$. We shall denote by ρ: $ML(n, R) \to GL(n, R)$ the map induced by the covering map $\tilde{\rho}$, and by χ: $ML(n, R) \to C$ the restriction of $\tilde{\chi}$ to $ML(n, R)$. A metalinear frame bundle for F is a principal $ML(n, R)$ fiber bundle $\widetilde{BF}$ over X together with a map τ: $\widetilde{BF} \to BF$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{BF} \times ML(n, R) & \longrightarrow & \widetilde{BF} \\ \downarrow \tau \times \rho & & \downarrow \rho \\ BF \times GL(n, R) & \longrightarrow & BF \end{array}$$

where the horizontal arrows denote the group actions. For each leaf Q of F, the restrictions of $\widetilde{BF}$ and BF to Q are principal fiber bundles with canonically defined flat connections (Śniatycki, 1975).

Let N be the fiber bundle over X associated to $\widetilde{BF}$ with standard fiber C on which $ML(n, R)$ acts by multiplication by $\chi(a)$. For each $\omega \in X$, an element $\nu_x \in N_x$ is a map from $\widetilde{BF}_x$ to C such that $\nu_x(ba) = \chi(a)\nu_x(b)$, for all $b \in \widetilde{BF}_x$ and $a \in ML(n, R)$. The Hermitian dual of N is a fiber bundle $N^\dagger$ over X associated to $\widetilde{BF}$ with standard fiber C on which $ML(n, R)$ acts by multiplication by

$\overline{\chi(a)}^{-1}$. For each $\mu_x \in N_x^\dagger$, $b \in \tilde{B}F_x$, and $a \in ML(n,R)$, $\mu_x(b) = \overline{\chi(a)}\mu_x(ba)$. The bundle $N^\dagger$ is called a bundle of half-forms relative to F. For each $x \in X$, there is a sesqulinear map $\langle \cdot | \cdot \rangle : N_x^\dagger \times N_x \to C$ such that $\langle \mu_x | \nu_x \rangle = \overline{\mu_x(b)}\nu_x(b)$ for any $b \in \tilde{B}F_x$.

For each integral manifold Q of F the restriction of BF to Q has a canonically defined flat connection. This defines flat connections in $N^\dagger | Q$ and $N|Q$. Therefore, we can differentiate covariantly sections of $N^\dagger$ or N in direction of vector fields in F. This covariant differentiation in direction of vector fields in F satisfies all the rules of genuine covariant differentiation. For any section μ of $N^\dagger$ and ν of N and any vector field ξ in F, we have $\xi(\langle \mu | \nu \rangle) = \langle \nabla_\xi \mu | \nu \rangle + \langle \mu | \nabla_\xi \nu \rangle$.

Since, for each leaf Q of F, the connection in $N|Q$ is flat, the parallel transpart in $N|Q$ defines a homomorphism $\alpha_{N|Q}: \pi_1(Q) \to C_*$.

Lemma 4.1. For each leaf Q of a complete locally trivial real polarization F, $\alpha_{N|Q}(\pi_1(Q)) \subset \{1, -1\}$. For any trivialization $\varphi: \eta^{-1}(U) \to U \times R^n/G$ of η and any $u \in G$, $\alpha_{N|Q}(\varphi_{Q*}^{-1}(u))$ is independent of Q in $\eta^{-1}(U)$.

Proof. Since $BF|Q$ is a trivial bundle, there is a subbundle $\tilde{B}^rF|Q$ of $\tilde{B}F|Q$ such that the inclusion map $\tilde{B}^rF|Q \to \tilde{B}F|Q$ gives a reduction of $ML(n,R)$ to the kernel of the covering map $\rho: ML(n,R) \to GL(n,R)$. Moreover, the connection in $\tilde{B}F|Q$ reduces to a connection in $\tilde{B}^rF|Q$. Hence, the holonomy groups of $\tilde{B}F|Q$ and $\tilde{B}^rF|Q$ coincide and they are subgroups of Ker ρ. Parallel transport in $N|Q$ defines a subgroup $\alpha_{N|Q}(\pi_1(Q))$ of C_* homomorphic to a subgroup of Ker ρ. Since ρ is a double covering, Ker ρ is isomorphic to Z_2 (the additive group of integers modulo 2), and therefore, $\alpha_{N|Q}(\pi_1(Q)) \subset \{1, -1\}$.

For any trivialization $\varphi: \eta^{-1}(U) \to U \times R^n/G$ and any $u \in G$, the map $g: U \to \{1, -1\}$ defined by $g(y) = \alpha_{N|Q}(\varphi_{Q*}^{-1}(u))$, where $Q = \eta^{-1}(y)$ is continuous. Therefore g is constant, which proves the second part of the Lemma.

Similarly, for each leaf Q of F, the connection in $N^\dagger|Q$ is flat, the parallel transport in $N^\dagger|Q$ defines a homomorphism from $\pi_1(Q)$ to C_*, and the image of this homomorphism is contained in $\{1, -1\}$. Since, for each $a \in$ Ker ρ, $\chi(a) = \overline{\chi(a)}^{-1}$, it follows that the homomorphism from $\pi_1(Q)$ to C_* induced by parallel transport in $N^\dagger|Q$ is the same as the homomorphism $\alpha_{N|Q}$ induced by the parallel transport in $N|Q$.

5. *Wave Functions*

The connection ∇ in the line bundle L and covariant differentiation of sections of N in direction of vector fields in F give rise to covariant differentiation of sections of $L \otimes N$ in direction of vector fields in F. For each section λ of L and each section ν of N, $\lambda \otimes \nu$ is a section of $L \otimes N$ and $\nabla_\xi(\lambda \otimes \nu) = (\nabla_\xi \lambda) \otimes \nu + \lambda \otimes \nabla_\xi \nu$ for all vector fields ξ in F. Let $D(L \otimes N)$ denote the space of C^∞ sections of $L \otimes N$ with compact supports, endowed with the standard topology, and $D'(L \otimes N)$ the space of continuous linear functionals on $D(L \otimes N)$. Elements of $D'(L \otimes N)$ are called generalized sections of $L \otimes N^\dagger$. The value of a generalized section ψ on a section $\sigma \in D(L \otimes N)$ is denoted by $\langle \psi, \sigma \rangle$. For each generalized section ψ of $L \otimes N^\dagger$ and each vector field ξ in F, the covariant derivative of ψ in direction ξ is a generalized section $\nabla_\xi \psi$ of $L \otimes N^\dagger$ defined by $\langle \nabla_\xi \psi, \sigma \rangle =$

$-\langle \psi, \nabla_\xi \sigma \rangle$, for all $\sigma \in D(L \otimes N)$. A generalized section ψ of $L \otimes N^\dagger$ is covariant constant along F if, for each Hamiltonian vector field ξ in $F, \nabla_\xi \psi = 0$. Generalized sections of $L \otimes N^\dagger$ covariant constant along F represent the quantum states of the physical system described by the classical phase space (X, ω), and they are called wave functions of the system in the representation given by the polarization F.

For each integral manifold Q of F the restriction of $L \otimes N$ to Q has a flat connection induced by the flat connection in $L|Q$ and the canonical flat connection in $N|Q$. The parallel transport of elements of $L \otimes N$ along loops in Q defines a homomorphism $\alpha_Q: \pi_1(Q) \to C_*$ such that, for each $\gamma \in \pi_1(Q)$, $\alpha_Q(\gamma) = \alpha_{L|Q}(\gamma)\alpha_{N|Q}(\gamma)$.

Theorem 5.1. If F is a complete locally trivial real polarization, then:

1. The supports of generalized sections of $L \otimes N^\dagger$ covariant constant along F are contained in the subset S of X consisting of all leaves Q of F for which the homomorphism $\alpha_Q: \pi_1(Q) \to C_*$ is trivial.
2. The projection Z of S to Y, $Z = \eta(S)$, is a submanifold of Y of codimension k equal to the rank of the fundamental group of a typical integral manifold of F.
3. S is a submanifold of X of codimension k, the characteristic distribution of $\omega|S$ coincides with the restriction to S of the distribution K defined in Theorem 2.1, that is

$$K|S = \{v \in TS \,|\, v \lrcorner \omega|S = 0\}$$

and there is a canonically defined density δ on S. For any Hamiltonian vector field ξ in F, the restriction of ξ to S leaves δ invariant.

Proof. 1. Let Q_0 be a leaf of F such that α_{Q_0} is not trivial. This means that there exists $\gamma \in \pi_1(Q_0)$ satisfying $\alpha_{Q_0}(\gamma) \neq 1$. By Theorem 2.1 there exists an open neighborhood $\eta^{-1}(U)$ of Q_0 and a vector field ζ on $\eta^{-1}(U)$ such that all integral curves c: $[0, 1] \to \eta^{-1}(U)$ of ζ are closed, and the homotopy class $[c_0]$ of any integral curve c: $[0, 1] \to Q_0$ of ζ is equal to γ. Without loss of generality we may assume that $\alpha_Q([c]) \neq 1$ for all Q in $\eta^{-1}(U)$ and all integral curves c: $[0, 1] \to Q$ of ζ.

Let ψ be a generalized section of $L \otimes N^\dagger$ covariant constant along F. We want to show that $\eta^{-1}(U) \cap \text{support } \psi = \emptyset$. Let σ be any smooth section of $L \otimes N$ with support contained in $\eta^{-1}(U)$. For each $t \in [0, 1]$, we denote by σ_t the section of $L \otimes N$ obtained from σ by the parallel transport along the integral curves of ζ, corresponding to the change of the parameters on these curves by the amount t. Since ψ is covariant constant along F and ζ is in F,

$$0 = \langle \nabla_\zeta \psi, \sigma_t \rangle = -\langle \psi, \nabla_\zeta \sigma_t \rangle = -(d/dt)\langle \psi, \sigma_t \rangle$$

Hence, $\langle \psi, \sigma_t \rangle$ is independent of t, and $\langle \psi, \sigma_1 \rangle = \langle \psi, \sigma_0 \rangle$. However, for each $x \in \eta^{-1}(U)$, $\sigma_1(x) = \alpha_{Q_x}([c_x])\sigma_0(x)$, where Q_x is the leaf of F through x and c_x: $[0, 1] \to Q_x$ is the integral curve of ζ originating at x. The function f: $\eta^{-1}(U) \to C_*$ defined by $f(x) = \alpha_{Q_x}([c_x])$ is smooth and $f(x) \neq 1$ for all $x \in \eta^{-1}(U)$. Therefore, for each smooth section σ of $L \otimes N$ with support in $\eta^{-1}(U)$, we have $\langle \psi, (f-1)\sigma \rangle = 0$, which implies that $\langle \psi, \sigma \rangle = 0$ for all smooth sections σ of $L \otimes N$ with supports in $\eta^{-1}(U)$. Hence, support $\psi \cap \eta^{-1}(U) = \emptyset$ which implies that support ψ is contained in the subset S of X consisting of all leaves Q of F for which α_Q is trivial.

2. Let $\varphi: \eta^{-1}(U) \to U \times R^n/G$ be a trivialization of η satisfying the conditions of Lemma 2.2 $u_1, \ldots, u_k$ the generators of G, and $\zeta_{u_1}, \ldots, \zeta_{u_k}$ the corresponding vector fields on $\eta^{-1}(U)$ defined in Theorem 2.1. For each $j = 1, \ldots, k, f_j: \eta^{-1}(U) \to C_*$ is a differentiable function of modulus 1 defined by $f_j(x) = \alpha_{Q_x}([c_{jx}])$, where Q_x is the leaf of F through x and $c_{jx}: [0,1] \to Q_x$ is the integral curve of ζ_{u_j} originating at x. Each of the functions f_j factors through the projection $\eta, f_j = g_j \circ \eta$, where $g_j: U \to C_*$ are smooth and $|g_j| = 1$, for all $j = 1, \ldots, k$. The set $Z \cap U$ is characterized by the conditions $g_j(y) = 1$ for all $j = 1, \ldots, k$ and all $y \in Z \cap U$. It suffices to show that the functions $g_1, \ldots, g_k$ are independent in U or, equivalently, that the functions $f_1, \ldots, f_k$ are independent. Let $c: [0, r] \to U, r > 0$, be a smooth curve in U. For each loop c_{jx}, where $\eta(x) = c(0)$, we have a homotopy $h_j: [0,r] \times [0,1] \to \eta^{-1}(U)$ defined by $h_j(s,t) = \varphi^{-1}(c(s), pr_2 \circ \varphi \circ c_{jx}(t))$ joining the loop c_{jx} to the loop $c_{jx'}$ where $x' = \eta^{-1}(c(r), pr_2 \circ \varphi(x))$. Therefore, we have

$$\begin{aligned} g_j(c(r)) - g_j(c(0)) &= \alpha_{Q_{x'}}([c_{jx'}]) - \alpha_{Q_x}([c_{jx}]) \\ &= \alpha_{N|Q_{x'}}([c_{jx'}])\alpha_{L|Q_{x'}}([c_{jx'}]) - \alpha_{N|Q_x}([c_{jx}])\alpha_{L|Q_x}([c_{jx}]) \\ &= \alpha_{N|Q_x}([c_{jx}])[\alpha_{L|Q_{x'}}([c_{jx'}]) - \alpha_{L|Q_x}([c_{jx}])] \\ &= \alpha_{N|Q_x}([c_{jx}]) \exp\left(-2\pi i \int_{Im h_j} \omega\right) \\ &= \alpha_{N|Q_x}([c_{jx}]) \exp\left[-2\pi i \int_0^r ds \int_0^1 dt h_j^*(\omega)((1,0),(0,1))\right] \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d}{dr} g_j \circ c(r)|_{r=0} &= -2\pi i \int_0^1 dt h_j^*(\omega)((1,0),(0,1))\alpha_{N|Q_x}([c_{jx}]) \\ &= -2\pi i \omega(v, \zeta_{u_j}(x))\alpha_{N|Q_x}([c_{jx}]) \end{aligned}$$

where v is any vector in $T_x X$ projecting to the tangent vector to c at $c(0)$. Therefore, $\zeta_{u_j}(x) \lrcorner \omega = \{2\pi i \alpha_{N|Q_x}([c_{jx}])\}^{-1} df_j$, for each $j = 1, \ldots, k$. Since the vector fields $\zeta_{u_1}, \ldots, \zeta_{u_k}$ are independent and ω is nonsingular, it follows that the functions $f_1, \ldots, f_k$ are independent. This implies that Z is a submanifold of Y of codimension k.

3. Since $S = \eta^{-1}(Z)$, it is clearly a submanifold of X of codimension k. Therefore, the characteristic distribution of $\omega|S$ is at most k-dimensional. For each $j = 1, \ldots, k, f_j(x) = 1$ for all $x \in S$. Hence, $v(f_j) = 0$ for all $v \in T_x S$, which implies that $(\zeta_{u_j} \lrcorner \omega)|S = 0$. Therefore, the distribution $K|S \cap \eta^{-1}(U)$ spanned by $\zeta_{u_1}|S, \ldots, \zeta_{u_k}|S$ is contained in the characteristic distribution of $\omega|S$. But

$\dim K|S = k$, and so $K|S$ coincides with the characteristic distribution of $\omega|S$. Finally, let V be an open subset of $\eta^{-1}(U)$ and $\xi_1, \ldots, \xi_{n-k}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-k}$ Hamiltonian vector fields on V such that $(\zeta_{u_1}, \ldots, \zeta_{u_k}, \xi_1, \ldots, \xi_{n-k})$ spans $F|V$, $(\zeta_{u_1}, \ldots, \zeta_{u_k}, \xi_1, \ldots, \xi_k, \tilde{\xi}_1, \ldots, \tilde{\xi}_k)|S$ spans $T(S \cap V)$, and $\omega(\xi_r, \tilde{\xi}_s) = \delta_{rs}$ and $\omega(\tilde{\xi}_s, \zeta_{u_j}) = 0$ for $r, s = 1, \ldots, n-k, j = 1, \ldots, k$. The density δ is the unique density which associates to the frame field $(\zeta_{u_1}, \ldots, \zeta_{u_k}, \xi_1, \ldots, \xi_k, \tilde{\xi}_1, \ldots, \tilde{\xi}_k)|S$ in S number 1. This condition defines δ since any other frame in $T(S \cap V)$ satisfying the same condition is related to the frame above by a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \in GL(2n-k, R)$$

where $|\mathrm{Det}\, A_{11}| = 1$, and $A_{22}A_{33}^T = 1$. Therefore $|\mathrm{Det}\, A| = |\mathrm{Det}\, A_{11}| \cdot |\mathrm{Det}\, A_{22}| \cdot |\mathrm{Det}\, A_{33}| = 1$. Further, the frame field $(\zeta_{u_1}, \ldots, \zeta_{u_k} \cdot \xi_1, \ldots, \xi_{n-k}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-k})$ is left invariant by any Hamiltonian vector field ξ on V with values in F. Therefore the density δ is left invariant by $\xi|S$.

6. *Scalar Product*

Let H_0 be the space of sections ψ of $(L \otimes N^\dagger)|S$ such that, for each leaf Q of F contained in S, $\psi|Q$ is covariant constant, and the projection to Z of the support of ψ is compact.

Proposition 6.1.

1. $\dim H_0 > 0$.
2. There is a canonical inclusion of H_0 into $D'(L \otimes N)$ such that elements of H_0 are generalized sections of $L \otimes N^\dagger$ covariant constant along F.
3. There is a canonically defined pre-Hilbert structure $(\cdot|\cdot)$ in H_0.

Proof. 1. Since η: $X \to Y$ is locally trivial, there exists a local section s: $U \to X$ of η such that U is contractible and $U \cap Z \neq 0$. Then, $s(U)$ is contractible and there exists a nowhere vanishing section $\tilde{s}$ of $(L \otimes N^\dagger)|s(U)$. Let ψ_s be the section of $(L \otimes N^\dagger)|S$ over $\eta^{-1}(U \cap Z)$ obtained from $\tilde{s}$ by parallel transport along leaves of F contained in $\eta^{-1}(U \cap Z)$. Let f be a smooth function on Z with a compact nonempty support in $U \cap Z$. Then, the local section $x \to f(\eta(x))$ $\psi_s(x)$ defined over $\eta^{-1}(U \cap Z)$ extends to a global section ψ: $S \to (L \otimes N^\dagger)|S$, which is not identically zero and belongs to H_0. Hence $\dim H_0 > 0$.

2. For any $\psi \in H_0$ and any $\sigma \in D(L \otimes N)$ there is a complex function $\langle \psi | \sigma \rangle$ on S defined as follows. For each $x \in S$, $\psi(x) = \lambda_x \otimes \mu_x$ and $\sigma(x) = \lambda'_x \otimes \nu_x$, where $\lambda_x \lambda'_x \in L_x$, $\mu_x \in N_x^\ddagger$ and $\nu_x \in N_x$. Then $\langle \psi(x) | \sigma(x) \rangle = \langle \lambda'_x | \lambda_x \rangle \langle \mu_x | \nu_x \rangle$, where $\langle \lambda'_x | \lambda_x \rangle$ is the Hermitian form in L evaluated on λ'_x and λ_x, and $\langle \mu_x | \nu_x \rangle$ is the value on (μ_x, ν_x) of the sesqulinear map $\langle \cdot | \cdot \rangle$: $N_x^\ddagger \times N_x \to C$ defined in Sec. 4. Clearly, this definition of $\langle \psi(x) | \sigma(x) \rangle$ is independent of the tensor product decomposition of $\psi(x)$ and $\sigma(x)$. Further, for each vector field ξ in F, we have $\xi \langle \psi | \sigma \rangle = \langle \psi | \nabla_\xi \sigma \rangle$. Each $\psi \in H_0$ defines a map $\langle \psi, \cdot \rangle$: $D(L \otimes N) \to C$ given

by $\langle \psi, \sigma \rangle = \int_S \langle \psi | \sigma \rangle \, \delta$, where δ is the density on S defined in Theorem 5.1. The map $\langle \psi, \cdot \rangle$ is linear and continuous, and $\langle \psi, \sigma \rangle = 0$ for all σ only if $\psi = 0$. Therefore, the association $\psi \mapsto \langle \psi, \cdot \rangle$ gives an imbedding of H_0 into $D'(L \otimes N)$. Moreover, for each Hamiltonian vector field ξ in F,

$$\langle \psi, \nabla_\xi \sigma \rangle = \int_S \langle \psi | \nabla_\xi \sigma \rangle \, \delta = \int_S \xi(\langle \psi | \sigma \rangle) \delta = 0$$

since ξ preserves δ and σ has compact support. Therefore, if we identify H_0 with its image in $D'(L \otimes N)$ under the map $\psi \mapsto \langle \psi, . \rangle$, each $\psi \in H_0$ is a generalized section of $L \otimes N^\dagger$ covariant constant along F.

3. Let ψ_1 and ψ_2 be elements of H_0. We shall define a density $\bar{\psi}_1 \cdot \psi_2$ on Z, dependening linearly on ψ_2 and antilinearly on ψ_1, and define the scalar product in H_0 by $(\psi_1 | \psi_2) = \int_Z \bar{\psi}_1 \cdot \psi_2$. Let y be any point in Z, and let $(\tilde{w}_1, \ldots, \tilde{w}_{n-k})$ be a basis in T_yZ. For any $x \in \eta^{-1}(y)$ there exists a basis in T_xX $(\tilde{u}_1, \ldots, \tilde{u}_k, v_1, \ldots, v_{n-k}, w_1, \ldots, w_{n-k}, t_1, \ldots, t_k)$ in T_xX such that $(\tilde{u}_1, \ldots, u_k)$ is a basis in K_x, $\kappa(\tilde{u}_1, \ldots, \tilde{u}_k) = 1$, $(\tilde{u}_1, \ldots, \tilde{u}_k, v_1, \ldots, v_{n-k})$ is a basis in F_x, $T\eta(w_r) = \tilde{w}_r$, $\omega(\tilde{u}_i, t_j) = \delta_{ij}$, $\omega(v_r, w_s) = \delta_{rs}$, $\omega(\tilde{u}_i, w_s) = \omega(v_r, t_j) = 0$, for $i, j = 1, \ldots, k$ and $r, s = 1, \ldots, n\text{-}k$. Let $b \in \tilde{B}F_x$ be such that $\tau(b) = (\tilde{u}_1, \ldots, \tilde{u}_k, v_1, \ldots, v_{n-k})$. If $\psi_1(x) = \lambda_{1x} \otimes \mu_{1x}$ and $\psi_2(x) = \lambda_{2x} \otimes \mu_{2x}$, for some $\lambda_{1x}, \lambda_{2x} \in L_x$ and $\mu_{1x}, \mu_{2x} \in N_x^\dagger$, we put $\bar{\psi}_1 \cdot \psi_2(\tilde{w}_1, \ldots, \tilde{w}_{n-k}) = \langle \lambda_{1x} | \lambda_{2x} \rangle \overline{\mu_{1x}(b)} \mu_{2x}(b)$. Clearly, this definition is independent of the choice of b projecting to $(\tilde{u}_1, \ldots, \tilde{u}_k, v_1, \ldots, v_{n-k})$ and the tensor product decomposition of $\psi_1(x)$ and $\psi_2(x)$. Any other basis in T_xX satisfying the same conditions will be related to the one we have used by a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix} \in GL(2n, R)$$

where $|\mathrm{Det}\, A_{11}| = 1$, $A_{11} A_{44}^T = 1$ and $A_{22} A_{33}^T = 1$, and $(\tilde{w}'_1, \ldots, \tilde{w}'_{n-k}) = (\tilde{w}_1, \ldots, \tilde{w}_{n-k}) A_{33}$ is a new basis in T_yZ. Therefore, an element $b' \in BF_x$ projecting onto the new basis in F_x will be related to b by $a \in ML(n, R)$ such that

$$\rho(a) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in GL(n, R)$$

Hence

$$\begin{aligned} \psi_1 \cdot \psi_2((\tilde{w}_1, \ldots, \tilde{w}_{n-k}) A_{33}) &= \bar{\psi}_1 \cdot \psi_2(\tilde{w}'_1, \ldots, \tilde{w}'_{n-k}) \\ &= \langle \lambda_{1x} | \lambda_{2x} \rangle \overline{\mu_{1x}(b')} \mu_{2x}(b') \\ &= \langle \lambda_{1x} | \lambda_{2x} \rangle \overline{\mu_{1x}(ba)} \mu_{2x}(ba) \\ &= \langle \lambda_{1x} | \lambda_{2x} \rangle \overline{\mu_1(b)} \mu_2(b) |\chi(a)|^{-2} \\ &= \langle \lambda_{1x} | \lambda_{2x} \rangle \overline{\mu_1(b)} \mu_2(b) |\mathrm{Det}\, A_{11} \cdot \mathrm{Det}\, A_{22}|^{-1} \\ &= \psi_1 \cdot \psi_2(\tilde{w}_1, \ldots, \tilde{w}_{n-k}) |\mathrm{Det}\, A_{33}| \end{aligned}$$

and $\bar{\psi}_1 \cdot \psi_2$ transforms as a density under the change of basis in T_yZ. Independence from $x \in \eta^{-1}(y)$ follows from the fact that ψ_1 and ψ_2 are covariant constant on $\eta^{-1}(y)$. Let x' be any point in $\eta^{-1}(y)$ and c: $[0, 1] \to \eta^{-1}(y)$ a curve joining x to x'. Along c we can choose autoparallel sections λ_1, λ_2 of L and μ_1, μ_2 of $N^\dagger$ such that $\psi_i(c(t)) = \lambda_i(c(t)) \otimes \mu_i(c(t))$, $i = 1, 2$, $t \in [0, 1]$, and a horizontal lift $\tilde{c}$ of c to BF. Then $\langle \lambda_1 | \lambda_2 \rangle \overline{\mu_1(\tilde{c})} \mu_2(\tilde{c})$ is a constant function along c. Hence the definition of $\bar{\psi}_1 \cdot \psi_2$ is independent of x in $\eta^{-1}(y)$. The scalar product $(\psi_1 | \psi_2) = \int_Z \bar{\psi}_1 \cdot \psi_2$ is thus well defined; and it clearly satisfies all the required properties.

The completion H of the pre-Hilbert space H_0 is the Hilbert space of wave functions in the representation given by polarization F.

Acknowledgments

The author is greatly indebted to K. Bleuler, R. J. Blattner, B. Kostant, J. Rawnsley, D. Simms, and K. Varadarajan for very helpful discussions on the problems studied in this paper.

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